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IV.

General Properties of certain Partial Differential Equations similar to those of Hydrodynamics.

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The following investigation was suggested by certain results which I obtained in a previous paper entitled "General Properties of the Equations of Steady Motion" which has been published by the United States Coast Survey. It was my intention in that paper to study steady motion only, but as I went on with the work, I perceived that many of the results were applicable to fluid motion in general, and that the whole subject admitted of a generalization which would present a number of interesting points.

In Vol. LIV. of "Crelle's Journal" there is a paper by Clebsch entitled "*Über eine allgemeine Transformation der hydrodynamischen Gleichungen,*" from which I have taken the proofs of two properties of a certain determinant; the proofs are also to be found in Baltzer's treatise and in Scott's treatise on determinants. I did not see Clebsch's paper until I had nearly completed mine, or I could have omitted one or two tedious processes that I have given by merely referring to Clebsch's article for the required demonstrations. It will perhaps be better, however, to leave this paper as it is, both because it is simpler and because the two papers are so unlike in their aims and methods of development, that, in order to merely refer to the article by Clebsch, it would be necessary for me to change my article throughout. Clebsch has in view merely a general transformation of the equations of hydrodynamics, while I have in view a general investigation of the "higher orders of motion" (vide Note at end) and of the properties of a certain set of Partial Differential Equations of the first order.

The present paper is divided into two parts, of which Part I. is herewith communicated to the Academy; Part II. is not yet completed. In Part II. it is my intention to generalize somewhat the methods and results of Part I. to make an application to the general theory of steady motion (n variables), and finally to apply all of the results of both parts to space of three dimensions, i. e. to the ordinary theory of hydrodynamics.

(Just here I wish to call attention to the Note at the end of this paper; it properly belongs in Part II., but as I wish it made public at the earliest opportunity, I have inserted it in Part I.)

In Part I. an account is given of a set of partial differential equations similar to those of hydrodynamics, but containing $n+1$ independent variables, or, if we may use the expression, the equations of fluid motion for space of n dimensions.

Denote by $x_1, x_2, x_3, \ldots . x_n, t$ the independent variables of the problem; by $u_1, u_2, \ldots . u_n, U$, certain functions of these variables. For a problem in hydrodynamics the function U is given by

$$U = V + \int \frac{dp}{\rho}, \tag{1}$$

V being the potential of external forces. The functions u are similar to the velocities in such an ordinary problem. Write

$$2\,q = u_1^2 + u_2^2 + \ldots . + u_n^2, \tag{2}$$

then

$$\frac{dq}{dx_i} = u_1 \frac{du_1}{dx_i} + u_2 \frac{du_2}{dx_i} + \ldots . + u_n \frac{du_n}{dx_i}. \tag{3}$$

Assume further

$$\xi_{12} = \frac{du_1}{dx_2} - \frac{du_2}{dx_1}, \qquad \xi_{13} = \frac{du_1}{dx_3} - \frac{du_3}{dx_1}, \text{ etc.,}$$

and generally

$$\xi_{jk} = \frac{du_j}{dx_k} - \frac{du_k}{dx_j}. \tag{4}$$

There will be of course

$$\frac{n\,(n-1)}{2}$$

of these quantities. The functions u are supposed to be connected by the relation

$$\frac{du_1}{dx_1} + \frac{du_2}{dx_2} + \ldots . + \frac{du_n}{dx_n} = 0. \tag{5}$$

Denoting now the operator

$$\frac{d^2}{dx_1^2} + \frac{d^2}{dx_2^2} + \ldots . + \frac{d^2}{dx_n^2}$$

by ∇, we have, on observing that $\xi_{jk} = -\,\xi_{kj}$,

$$\begin{array}{l} \frac{d\xi_{12}}{dx_2} + \frac{d\xi_{13}}{dx_3} + \ldots + \frac{d\xi_{1n}}{dx_n} = \nabla u_1, \\ \frac{d\xi_{21}}{dx_1} + \frac{d\xi_{23}}{dx_3} + \ldots + \frac{d\xi_{2n}}{dx_n} = \nabla u_2, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \frac{d\xi_{n1}}{dx_1} + \frac{d\xi_{n2}}{dx_2} + \ldots + \frac{d\xi_{n,n-1}}{dx_{n-1}} = \nabla u_n. \end{array} \qquad (6)$$

Between the three quantities ξ_{lm}, ξ_{mn}, ξ_{nl} there exists for all values of l, m, n the relation

$$\frac{d\xi_{mn}}{dx_l} + \frac{d\xi_{nl}}{dx_m} + \frac{d\xi_{lm}}{dx_n} = 0. \qquad (7)$$

This relation holds throughout the three-dimensional space determined by the directions x_l, x_m, x_n; and as these are perfectly arbitrary, it is clear that a like relation holds in each of the Euclidean spaces determined by the directions x taken in groups of three. There does not appear to be any simple expression of this form connecting all of the ξ-functions.

Consider for a moment the differential expression

$$u_1 dx_1 + u_2 dx_2 + \ldots + u_n dx_n; \qquad (\alpha)$$

the conditions that this shall be an exact differential are $\xi_{12} = \xi_{13} = \ldots = \xi_{n,n-1} = 0$; or, as these may all be written,

$$\left\| \begin{array}{cccc} \frac{d}{dx_1}, & \frac{d}{dx_2}, & \ldots & \frac{d}{dx_n} \\ u_1, & u_2, & \ldots & u_n \end{array} \right\| = 0. \qquad (8)$$

If this quantity is an exact differential, say $d\phi$, the function ϕ must by virtue of (5) satisfy the differential equation

$$\nabla\phi = 0.$$

If (α) can be made an exact differential by choosing a proper integrating factor, we must have

$$u_i\xi_{jk} + u_j\xi_{ki} + u_k\xi_{ij} = 0 \qquad (9)$$

for all values of i, j, k; or, as these may be written,

$$\left\| \begin{array}{cccc} u_1, & u_2, & \ldots & u_n \\ \frac{d}{dx_1}, & \frac{d}{dx_2}, & \ldots & \frac{d}{dx_n} \\ u_1, & u_2, & \ldots & u_n \end{array} \right\| = 0. \qquad (10)$$

In a paper by Professor H. W. Lloyd Tanner "On the Transformation of a Linear Differential Expression," contained in the Quarterly Journal of Mathematics for December, 1878, it is proved that

$$u_1dx_1 + u_2dx_2 + \ldots\ldots + u_ndx_n$$

is reducible to the form

$$dv_1 + w_2dv_2 + w_3dv_3 + \ldots\ldots + w_rdv_r, \tag{β}$$

if, and only if,

$$\left\| \begin{matrix} \dfrac{d}{dx_1}, & \dfrac{d}{dx_2}, & \ldots\ldots & \dfrac{d}{dx_n} \\ u_1, & u_2, & \ldots\ldots & u_n \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \dfrac{d}{dx_1}, & \dfrac{d}{dx_2}, & \ldots\ldots & \dfrac{d}{dx_n} \\ u_1, & u_2, & \ldots\ldots & u_n \end{matrix} \right\| = 0, \tag{11}$$

where there are $2r$ rows; and that the same quantity is reducible to the form

$$w_1dv_1 + w_2dv_2 + \ldots\ldots + w_rdv_r,$$

if, and only if,

$$\left\| \begin{matrix} u_1, & u_2, & \ldots\ldots & u_n \\ \dfrac{d}{dx_1}, & \dfrac{d}{dx_2}, & \ldots\ldots & \dfrac{d}{dx_n} \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \dfrac{d}{dx_1}, & \dfrac{d}{dx_2}, & \ldots\ldots & \dfrac{d}{dx_n} \\ u_1, & u_2, & \ldots\ldots & u_n \end{matrix} \right\| = 0, \tag{12}$$

where there are $2r + 1$ rows. In both of these cases the functions w and v, though functions of $x_1, x_2, \ldots\ldots x_n$, are not mutually dependent. In the case in hand it will be shown that the expression (α) can always be reduced to the form (β) when $r = n + 1$. The case when $u_1dx_1 + u_2dx_2 + \ldots\ldots + u_ndx_2$ is an exact differential $d\phi$ gives for ϕ, as already remarked,

$$\frac{d^2\phi}{dx_1^2} + \frac{d^2\phi}{dx_2^2} + \ldots\ldots + \frac{d^2\phi}{dx_n^2} = 0. \tag{13}$$

I will give here, for convenience of reference, the transformation of this equation in polar coördinates. Some of the properties of the function ϕ will be found in another part of the paper. Denote by $r, \alpha_1, \alpha_2, \ldots\ldots \alpha_{n-1}$ the polar coördinates of a point in space of n dimensions. Then write

$$\begin{aligned}
x_1 &= r \cos a_1, \\
x_2 &= r \sin a_1 \cos a_2, \\
x_3 &= r \sin a_1 \sin a_2 \cos a_3, \\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
x_i &= r \sin a_1 \sin a_2 \ldots . \sin a_{i-1} \cos a_i \\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
x_n &= r \sin a_1 \sin a_2 \ldots . \sin a_{n-1}.
\end{aligned} \tag{14}$$

From these we have

$$r = (x_1^2 + x_2^2 + \ldots . + x_n^2)^{\frac{1}{2}},$$

$$\begin{array}{ll}
\cos a_1 = \dfrac{x_1}{r}, & \sin a_1 = \dfrac{\sqrt{r^2 - x_1^2}}{r}, \\[2ex]
\cos a_2 = \dfrac{x_2}{\sqrt{r^2 - x_1^2}}, & \sin a_2 = \dfrac{\sqrt{r^2 - (x_1^2 + x_2^2)}}{\sqrt{r^2 - x_1^2}}, \\[2ex]
\cos a_3 = \dfrac{x_3}{\sqrt{r^2 - (x_1^2 + x_2^2)}}, & \sin a_3 = \dfrac{\sqrt{r^2 - (x_1^2 + x_2^2 + x_3^2)}}{\sqrt{r^2 - (x_1^2 + x_2^2)}}, \\[2ex]
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\[2ex]
\cos a_i = \dfrac{x_i}{\sqrt{r^2 - \sum\limits_{k=1}^{k=i-1} x_k^2}}, & \sin a_i = \dfrac{\sqrt{r^2 - \sum\limits_{k=1}^{k=i} x_k^2}}{\sqrt{r^2 - \sum\limits_{k=1}^{k=i-1} x_k^2}}, \\[2ex]
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \\[2ex]
\tan a_{n-1} = \dfrac{x_n}{x_{n-1}}, & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot
\end{array} \tag{15}$$

Now we have at once (writing for the moment $a_0 = r$)

$$\begin{aligned}
\nabla^2\phi = {} & \frac{d^2\phi}{da_0^2} \Sigma \left(\frac{da_0}{dx_k}\right)^2 + \frac{d^2\phi}{da_1^2} \Sigma \left(\frac{da_1}{dx_k}\right)^2 + \ldots . + 2\frac{d^2\phi}{da_0 da_1} \Sigma \frac{da_0}{dx_k}\frac{da_1}{dx_k} + 2\frac{d^2\phi}{da_0 da_2} \Sigma \frac{da_0}{dx_k}\frac{da_2}{dx_k} + \ldots . \\
& + \frac{d\phi}{da_0} \nabla^2 a_0 + \frac{d\phi}{da_1} \nabla^2 a_1 + \frac{d\phi}{da_2} \nabla^2 a_2 + \ldots .
\end{aligned} \tag{16}$$

The summations in this formula are for all values of k from 1 to i. By performing the necessary operations we find quite readily

$$\Sigma \left(\frac{da_0}{dx_k}\right)^2 = 1, \qquad \Sigma \left(\frac{da_1}{dx_k}\right)^2 = \frac{1}{a_0^2}, \qquad \Sigma \left(\frac{da_2}{dx_k}\right)^2 = \frac{1}{a_0^2 \sin^2 a_1},$$

$$\sum_k \left(\frac{da_i}{dx_k}\right)^2 = \frac{1}{a_0^2 \Pi \sin^2 a_j}, \qquad j = 1, 2, \ldots . i - 1,$$

$$\sum_k \frac{d\alpha_i}{dx_k}\,\frac{d\alpha_{i+1}}{dx_k} = 0, \qquad i = 0, 1, 2, \ldots . \, n-1,$$

$$\nabla^2\alpha_0 = \frac{n-1}{\alpha_0}, \qquad \nabla^2\alpha_1 = \frac{(n-2)\cos\alpha_1}{\alpha_0{}^2\sin\alpha_1}, \qquad \nabla^2\alpha_2 = \frac{(n-3)\cos\alpha_2}{\alpha_0{}^2\sin\alpha_2\sin^2\alpha_1},$$

$$\nabla^2\alpha_j = \frac{[n-(i+1)]\cos\alpha_i}{\sin\alpha_i \prod\limits_{j=1}^{j=i-1}\sin^2\alpha_j}, \qquad \nabla^2\alpha_{n-1} = 0.$$

Substituting in $\nabla^2\phi$, replacing α_0 by r, and multiplying through by r^2 we have

$$\begin{aligned}\Delta^2\phi = {} & r^2\frac{d^2\phi}{dr^2} + \frac{d^2\phi}{d\alpha_1{}^2} + \frac{1}{\sin^2\alpha_1}\frac{d^2\phi}{d\alpha_2{}^2} + \ldots . + \frac{1}{\prod\limits_{j=1}^{j=i-1}\sin^2\alpha_j}\frac{d^2\phi}{d\alpha_i{}^2} + \ldots . + \frac{1}{\prod\limits_{j=1}^{j=n-2}\sin^2\alpha_j}\frac{d^2\phi}{d\alpha_{n-1}^2} \\ & + r\frac{d\phi}{dr}(n-1) + \frac{d\phi}{d\alpha_1}\frac{(n-2)\cos\alpha_1}{\sin\alpha_1} + \frac{d\phi}{d\alpha_2}\frac{(n-3)\cos\alpha_2}{\sin\alpha_2\sin^2\alpha_1} + \ldots . + \frac{d\phi}{d\alpha_i}\frac{[n-(i+1)]\cos\alpha_i}{\sin\alpha_i\prod\limits_{j=1}^{j=i-1}\sin^2\alpha_j} \\ & + \ldots . \end{aligned} \tag{16}$$

or, arranging a little differently,

$$\begin{aligned} & \left[r\frac{d^2(r\phi)}{dr^2} + \frac{1}{\sin\alpha_1}\frac{d}{d\alpha_1}\left(\sin\alpha_1\frac{d\phi}{d\alpha_1}\right) + \frac{1}{\sin\alpha_2 \,.\, \sin^2\alpha_1}\frac{d}{d\alpha_2}\left(\sin\alpha_2\frac{d\phi}{d\alpha_2}\right) + \ldots . \right. \\ & \left. + \frac{1}{\sin\alpha_i\prod\limits_{j=1}^{j=i-1}\sin^2\alpha_j}\frac{d}{d\alpha_i}\left(\sin\alpha_i\frac{d\phi}{d\alpha_i}\right) + \ldots . + \frac{1}{\prod\limits_{j=1}^{j=n-2}\sin^2\alpha_j}\frac{d^2\phi}{d\alpha_{n-1}}\right] \\ & + \left[(n-3)\,r\frac{d\phi}{dr} + (n-3)\frac{\cos\alpha_1}{\sin\alpha_1}\frac{d\phi}{d\alpha_1} + (n-4)\frac{\cos\alpha_2}{\sin\alpha_2}\cdot\frac{1}{\sin^2\alpha_1}\frac{d\phi}{d\alpha_2} + \ldots . \right. \\ & \left. + [n-(i+2)]\frac{\cos\alpha_i}{\sin\alpha_i}\cdot\frac{1}{\prod\limits_{j=1}^{j=i-1}\sin^2\alpha_j}\frac{d\phi}{d\alpha_i} + \ldots + \frac{\cos\alpha_{n-2}}{\sin\alpha_{n-2}}\cdot\frac{1}{\prod\limits_{j=1}^{j=n-2}\sin^2\alpha_j}\cdot\frac{d\phi}{d\alpha_{n-2}}\right] = 0. \end{aligned} \tag{17}$$

Denote by T the region of all values of $x_1, x_2 \ldots . x_n$ for which

$$x_1{}^2 + x_2{}^2 + \ldots . + x_n{}^2 = R^2 \tag{18}$$

i. e. R is the superior limit of r. Denote by S the boundary of T; then S signifies the totality of all systems of values for which

$$x_1{}^2 + x_2{}^2 + \ldots . + x_n{}^2 = R_2. \tag{19}$$

The equation $\nabla^2\phi = 0$ holds throughout the entire region T and right up to the boundary S. Then it can be proved just as in the case of three variables that there is one and only one real function ϕ of the variables x, that for the whole region T is single valued and continuous, and at the boundary S coincides with an arbitrary function spread out continuously over S. The first and second differential coefficients of ϕ will be continuous, and the latter will satisfy the equation $\nabla^2\phi = 0$. Denote by dT the element of the region T; then

$$dT = dx_1 dx_2 dx_3 \ldots . dx_n = r^{n-1} \Big(\prod_{k=1}^{k=n-2} \sin^{n-k-1} \alpha_k \, d\alpha_k \Big) \, d\phi^{n-1} dr. \tag{20}$$

The element of the boundary S is dS, and this is readily found to be given by

$$dS = R^{n-1} \Big(\prod_{k=1}^{k=n-2} \sin^{n-k-1} \alpha'_k \, d\alpha'_k \Big) \, d\alpha'_{n-1} \tag{21}$$

($R, \alpha'_1, \alpha'_2, \ldots . \alpha'_{n-1}$ being the limits of $r, \alpha_1, \alpha_2, \ldots . \alpha_{n-1}$)

Introducing now an angle δ defined by

$$\cos\delta = \cos\alpha_1 \cos\alpha'_1 + \sum_{k=1}^{k=n-1} \Big\{ \cos\alpha_{k+1} \cos\alpha'_{k+1} \prod_{j=1}^{j=k} \sin\alpha_j \sin\alpha'_j \Big\} + \prod_{k=1}^{k=n-1} \sin\alpha_k \sin\alpha'_k \tag{22}$$

The introduction of this angle δ enables us to obtain quite easily

$$\phi = \frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{\frac{n}{2}}} \int_1\!\!\int_2 \ldots . \int_{n-1} \frac{\phi_1 (R^2 - r^2)\, dS}{R\,(R^2 + r^2 - 2\,Rr\cos\delta)^{\frac{n}{2}}} \tag{23}$$

ϕ_1 denoting the arbitrarily chosen surface value of ϕ.

Further consideration of this function ϕ will be deferred for the present, as a full investigation of its properties would require the development of the theory of spherical harmonics for a space of n dimensions. Equation (17) will afford the means of obtaining some of the more elementary of these properties.

We will resume here the consideration of the quantities ξ_{jk}. Observe that if we denote the quantities of the left-hand sides of equations (6) by a single letter, say by $\eta_1, \eta_2, \ldots .$ that we have identically

$$\frac{d\eta_1}{dx_1} + \frac{d\eta_2}{dx_2} + \ldots . + \frac{d\eta_i}{dx_i} = 0. \tag{24}$$

If now from the η's we form quantities similar to the ξ's and call them $\xi^{(1)}_{12}, \xi^{(1)}_{13} \ldots .$ we shall have a set of equations of the same form as (6) with ξ replaced by $\xi^{(1)}$ and u replaced by η. Calling the left-hand members of this new set $\eta^{(1)}_1, \eta^{(1)}_2, \ldots .$, and again performing the same operations, we get equations similar to (6) with the ξ replaced by $\xi^{(2)}$, and for the right-hand members $\nabla^{2.2}u$, $\nabla^{2.2}u_2$, etc., where $\nabla^{2.2}$ means that the operation ∇^2 has been performed twice. This process can of course go on indefinitely or until we come to such values of the η-functions that

$$\eta^{(k)}_1 dx_1 + \eta^{(k)}_2 dx_2 + \ldots . + \eta^{(k)}_i dx_i$$

is an exact differential. Instead of treating these functions $u_1, u_2 \ldots . u_i$ altogether we may consider them combined in groups of three each; i. e. consider the motion in

$$\frac{n(n-1)(n-2)}{6}$$

flat spaces of three dimensions instead of the motion in one space of n dimensions. Equation (7) shows the relation that exists between the rotation components ξ_{jk} in such a space; of course this relation holds for all of the Euclidean spaces which can be formed from the space of n dimensions, and therefore a similar relation exists for the n-dimensional space formed by suming the expression on the left-hand side of (7) for all values of l, m, n. Write for brevity

$$\frac{n(n-1)(n-2)}{6} = I.$$

If the motion in any one of the I three dimensional spaces is rotational, it is of course rotational for the space of n dimensions.

For the space containing x_l, x_m, x_n, we have

$$\begin{aligned}
\frac{d\xi_{lm}}{dx_l} - \frac{d\xi_{mn}}{dx_n} &= \frac{d}{dx_m}\left(\frac{dn_l}{dx_l} + \frac{dn_m}{dx_m} + \frac{dn_n}{dx_n}\right) - \nabla^2 u_m \\
\frac{d\xi_{mn}}{dx_m} - \frac{d\xi_{nl}}{dx_l} &= \frac{d}{dx_n}\left(\frac{dn_l}{dx_l} + \frac{dn_m}{dx_m} + \frac{dn_n}{dx_n}\right) - \nabla^2 u_n \qquad (25) \\
\frac{d\xi_{nl}}{dx_n} - \frac{d\xi_{lm}}{dx_m} &= \frac{d}{dx_l}\left(\frac{dn_l}{dx_l} + \frac{dn_m}{dx_m} + \frac{dn_n}{dx_n}\right) - \nabla^2 u_l
\end{aligned}$$

where ∇^2 is written for

$$\frac{d^2}{dx_l^2} + \frac{d^2}{dx_m^2} + \frac{d^2}{dx_n^2}.$$

For ordinary space the quantities in parentheses vanish and these reduce to well-known forms. Write them in the forms

$$\begin{aligned}
\eta_l &= \frac{dR}{dx_l} - \nabla^2 u_l \\
\eta_m &= \frac{dR}{dx_m} - \nabla^2 u_m \qquad (26) \\
\eta_n &= \frac{dR}{dx_n} - \nabla^2 u_n
\end{aligned}$$

Form now the quantities

$$\eta_l' = \frac{d\eta_n}{dx_m} - \frac{d\eta_m}{dx_n} \qquad \eta_m' = \text{etc.} \qquad (27)$$

These operations cause the quantity R to disappear and leave us

$$\begin{aligned}\eta_l' &= \nabla^2 \xi_{mn}\\ \eta_m' &= \nabla^2 \xi_{nl}\\ \eta_n' &= \nabla^2 \xi_{lm}.\end{aligned}\tag{28}$$

Perform the same operations on the η' functions, and call the results η_l'', η_m'', η_n''.

$$\begin{aligned}\eta_l'' &= \nabla^2 \frac{dR}{dx_l} - \nabla^{2,2} u_l\\ \eta_m'' &= \nabla^2 \frac{dR}{dx_m} - \nabla^{2,2} u_m\\ \eta_n'' &= \nabla^2 \frac{dR}{dx_n} - \nabla^{2,2} u_n.\end{aligned}\tag{29}$$

This operation can be carried on indefinitely, and we shall have in each case of an η with an odd number of accents that the quantity R, or the expansion, as we may call it, disappears, while for the η's with an even number of accents it appears in the form

$$\nabla^{2k} \frac{dR}{dx}$$

k being here an odd number equal to the number of accents diminished by unity. For an incompressible fluid we have therefore a set of very simple formulas giving the relations between these various orders of motion.

We shall now introduce the equations of motion ; these are

$$\begin{aligned}\frac{dU}{dx_1} &= \frac{du_1}{dt} + u_1 \frac{du_1}{dx_1} + u_2 \frac{du_1}{dx_2} + \cdots + u_n \frac{du_1}{dx_n}\\ \frac{dU}{dx_2} &= \frac{du_2}{dt} + u_1 \frac{du_2}{dx_1} + u_2 \frac{du_2}{dx_2} + \cdots + u_n \frac{du_2}{dx_n}\\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\\ \frac{dU}{dx_n} &= \frac{du_n}{dt} + u_1 \frac{du_n}{dx_1} + u_2 \frac{du_n}{dx_2} + \cdots + u_n \frac{du_n}{dx_n}.\end{aligned}\tag{30}$$

Introducing the quantities ξ and q and writing $U - q = W$, these equations may be written

$$\begin{aligned}\frac{dW}{dx_1} &= \frac{du_1}{dt} + u_2 \xi_{12} + u_3 \xi_{13} + \cdots + u_n \xi_{1n}\\ \frac{dW}{dx_2} &= \frac{du_2}{dt} - u_1 \xi_{12} + u_3 \xi_{23} + \cdots + u_n \xi_{2n}\\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\\ \frac{dW}{dx_n} &= \frac{du_n}{dt} - u_1 \xi_{1n} - u_2 \xi_{2n} - \cdots - u_{n-1} \xi_{n-1,n}.\end{aligned}\tag{31}$$

For steady motion the terms $\frac{du}{dt}$ vanish, and then multiplying these equations by $u_1, u_2 \ldots u_n$ respectively and adding, we have

$$u_1 \frac{dW}{dx_1} + u_2 \frac{dW}{dx_2} + \cdots + u_n \frac{dW}{dx_n} = 0 \tag{32}$$

Equations (31) may be written in the form

$$\frac{dW}{dx_k} = \frac{du_k}{dt} + \Xi_k \tag{33}$$

Then, by taking equation (5) into account, we have for the determination of W

$$\nabla^2 W = \sum_{k=1}^{k=n} \frac{d\Xi_k}{dx_k} \tag{34}$$

Take now any two of equations (10), say those indicated by the subscripts j and k; differentiate the j-equation for x_k and the k-equation for x_j, and subtract the second result from the first; continue the process until all of equations (10) have been combined two at a time by this process; we thus arrive at a group of $\frac{n(n-1)}{2}$ equations of the form

$$\begin{aligned}\frac{d\xi_{12}}{dt} &+ u_1 \frac{d\xi_{12}}{dx_1} + u_2 \frac{d\xi_{12}}{dx_2} + \cdots + u_n \frac{d\xi_{12}}{dx_n} \\ &+ \xi_{12}\left(\frac{du_1}{dx_1} + \frac{du_2}{dx_2}\right) + \xi_{13}\frac{du_3}{dx_2} + \xi_{14}\frac{du_4}{dx_2} + \cdots + \xi_n \frac{du_n}{dx_2} \\ &- \xi_{23}\frac{du_3}{dx_1} - \xi_{24}\frac{du_4}{dx_1} + \cdots \xi_{2n}\frac{du_n}{dx_n} = 0\end{aligned}$$

or generally

$$\begin{aligned}\frac{d\xi_{jk}}{dt} &+ u_1 \frac{d\xi_{jk}}{dx_1} + u_2 \frac{d\xi_{jk}}{dx_2} + \cdots + u_n \frac{d\xi_{jk}}{dx_n} + \xi_{jk}\left(\frac{du_j}{dx_j} + \frac{du_k}{dx_k}\right) \\ &+ \xi_{1k}\frac{du_1}{dx_j} + \xi_{2k}\frac{du_2}{dx_j} + \cdots + \xi_{jn}\frac{du_n}{dx_j} \\ &- \xi_{1j}\frac{du_1}{dx_k} - \xi_{2j}\frac{du_2}{dx_k} - \cdots - \xi_{ki}\frac{du_n}{dx_k} = 0.\end{aligned} \tag{35}$$

Since $\xi_{jj} = \xi_{kk} = 0$ and $\xi_{jk} = -\xi_{kj}$, we may write this equation in the form

$$\frac{d\xi_{jk}}{dt} + u_1 \frac{d\xi_{jk}}{dx_1} + u_2 \frac{d\xi_{jk}}{dx_2} + \cdots + u_n \frac{d\xi_{jk}}{dx_n} + \sum_{r=1}^{r=n} \begin{Vmatrix} \xi_{rk} & \xi_{rj} \\ \frac{du_r}{dx_k} & \frac{du_r}{dx_j} \end{Vmatrix} = 0 \tag{36}$$

These equations give the vortex motion in the n-dimensional space; for $n = 3$, they obviously become the ordinary equations for vortex motion.

The equations giving the motion of a fluid particle are, of course,

$$\frac{dx_1}{dt}=u_1,\frac{dx_2}{dt}=u_2\ldots\frac{dx_n}{dt}=u_n. \tag{37}$$

Writing for brevity

$$\frac{D}{dt}=\frac{d}{dt}+u_1\frac{d}{dx_1}+u_2\frac{d}{dx_2}+\cdots u_n\frac{d}{dx_n}$$

equations (30) become

$$\frac{dU}{dx_1}=\frac{Du_1}{dt},\qquad\frac{dU}{dx_2}=\frac{Du_2}{dt},\text{ etc.} \tag{38}$$

Multiplying these by $dx_1, dx_2\ldots$ and adding, we have readily

$$dU+dq=\frac{D}{dt}(u_1dx_1+u_2dx_2+\cdots u_ndx_n). \tag{39}$$

Integrating from a point α to another β, we have

$$\frac{D}{dt}\int_\alpha^\beta(u_1dx_1+u_2dx_2+\cdots u_ndx_n)=[U+q]_\alpha^\beta. \tag{40}$$

If α coincide with β, i. e. if we integrate around a closed curve,

$$\frac{D}{dt}\int(u_1dx_1+u_2dx_2+\cdots+u_ndx_n)=0, \tag{41}$$

a theorem corresponding to that in hydrodynamics which states that the circulation in any closed curve moving with the fluid is independent of the time. As the principal object of this paper is to carry on and generalize the investigations begun in the paper on steady motion previously referred to, I shall leave for the present any further examination of the properties of the quantities dealt with so far and begin from a rather more general point of view. In steady motion a certain Jacobian appeared which entered into all of the work in a curious manner, this Jacobian was found from the functions θ_1, θ_2, θ_3, i. e.

$$\begin{Vmatrix}\frac{d\theta_1}{dx}, & \frac{d\theta_1}{dy}, & \frac{d\theta_1}{dz}\\ \frac{d\theta_2}{dx}, & \frac{d\theta_2}{dy}, & \frac{d\theta_2}{dz}\\ \frac{d\theta_3}{dx}, & \frac{d\theta_3}{dy}, & \frac{d\theta_3}{dz}\end{Vmatrix}=J. \tag{42}$$

In the present more general problem we may introduce a set of functions

$$\theta,\theta_1,\theta_2\ldots\theta_{n-1}$$

of $x_1, x_2 \ldots . x_n, t$, denote the Jacobian of these by J, i. e.

$$\left\| \begin{array}{l} \frac{d\theta}{dx_1}, \frac{d\theta}{dx_2}, \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \;, \frac{d\theta}{dx_n} \\ \frac{d\theta_1}{dx_1}, \frac{d\theta_1}{dx_2}, \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \;, \frac{d\theta_1}{dx_n} \\ \frac{d\theta_{n-1}}{dx_1}, \frac{d\theta_{n-1}}{dx_2}, \ldots \ldots, \frac{d\theta_{n-1}}{dx_n} \end{array} \right\| = J. \qquad (43)$$

The minors corresponding to elements in the first row may be denoted by

$$A_{11}, A_{12}, \ldots \ldots, A_{1n},$$

and in general for the minors to the elements in the i^{th} row

$$A_{i1}, A_{i2}, \ldots \ldots, A_{in}.$$

Of course we have now

$$J = A_{i1}\frac{d\theta_{i-1}}{dx_1} + A_{i2}\frac{d\theta_{i-1}}{dx_2} + \ldots . + A_{in}\frac{d\theta_{i-1}}{dx_n}. \qquad (44)$$

The minors A satisfy identically the equations

$$\frac{dA_{i1}}{dx_1} + \frac{dA_{i2}}{dx_2} + \ldots . + \frac{dA_{1n}}{dx_n} = 0, \quad \text{etc.} \qquad (45)$$

If we substitute $A_{11}\ A_{12} \ldots . A_{1n}$ for $u_1, u_2 \ldots . u_n$ in equations (38) we obtain at once

$$dU = \sum_{j=1}^{j=n} \frac{DA_{1j}}{dt}\, dx_j \qquad (46)$$

or

$$dU = \sum_{j=1}^{j=n} \frac{dA_{1j}}{dt}\, dx_j + \sum_{k=1}^{k=n}\sum_{j=1}^{j=n} A_{1k}\frac{dA_{1j}}{dx_k}\, dx_j. \qquad (47)$$

Introducing the quantity $2\,q$ which is now

$$= A_{11}^2 + A_{12}^2 + \ldots . + A_{1n}^2$$

(47) becomes

$$d(U-q) = \sum_{j=1}^{j=n} \frac{dA_{1j}}{dt}\, dx_j + \sum_{k=1}^{k=n}\sum_{j=1}^{j=n} A_{1k} \left\| \begin{array}{cc} \frac{d}{dx_k} & \frac{d}{dx_j} \\ A_{1k} & A_{1j} \end{array} \right\| dx_j. \qquad (48)$$

For brevity write

$$K_j = \sum_{k=1}^{k=n} A_{1k} \left\| \begin{array}{cc} \frac{d}{dx_k} & \frac{d}{dx_j} \\ A_{1k} & A_{1j} \end{array} \right\| \qquad (49)$$

Assume a set of functions $n-1$ in number

$$\Phi_1, \Phi_2, \ldots . \Phi_{n-1}$$

and write

$$
\begin{aligned}
K_1 &= \Phi_1 \frac{d\theta_1}{dx_1} + \Phi_2 \frac{d\theta_2}{dx_1} + \ldots\ldots + \Phi_{n-1} \frac{d\theta_n}{dx_1}, \\
K_2 &= \Phi_1 \frac{d\theta_1}{dx_2} + \Phi_2 \frac{d\theta_2}{dx_2} + \ldots\ldots + \Phi_{n-1} \frac{d\theta_n}{dx_2}, \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
K_n &= \Phi_1 \frac{d\theta_1}{dx_n} + \Phi_2 \frac{d\theta_2}{dx_n} + \ldots\ldots + \Phi_{n-1} \frac{d\theta_n}{dx_n}.
\end{aligned} \tag{50}
$$

Multiply each of these equations by $A_{11}, A_{12}, \ldots\ldots A_{1n}$ respectively and add: this gives

$$A_{11} K_1 + A_{12} K_2 + \ldots\ldots + A_{1n} K_n = 0 \tag{51}$$

a necessary condition, as is easily seen; for if we multiply each K as given in (49) by the minor A having the same member for its second subscript and add, there results

$$A_{11} K_1 + A_{12} K_2 + \ldots\ldots + A_{1n} K_n = \sum_{k=1}^{k=n} \sum_{j=1}^{j=n} A_{1j} A_{1k} \left(\frac{dA_{1j}}{dx_k} - \frac{dA_{1k}}{dx_j} \right). \tag{52}$$

An interchange of j with k in this should not alter the value of the right-hand member of the equation, but it obviously does alter it to the extent of changing its sign, and the quantity is therefore equal zero, or

$$A_{11} K_1 + A_{12} K_2 + \ldots\ldots A_{1n} K_n = 0.$$

For the determination of the functions Φ we may proceed as follows. Denote the minors of A^{1h} with respect to the element in the $i + 1^{\text{th}}$ row and k^{th} column by A^{ik}_{1h}; then the quantity

$$\sum_{i=1}^{i=n-1} \frac{d\theta_i}{dx_j} A^{ik}_{1h}$$

represents the determinant found from A_{1h} by changing the column of elements containing differential coefficients with respect to x_k into a column containing differential coefficients with respect to x_j. In general then this determinant must vanish, as there will exist in it two columns containing the derivatives of the functions θ with respect to x_j. If $j = k$ the determinant becomes simply A_{1h}. In case $j = h$ the determinant contains no derivatives with respect to x_j and for that case

$$\sum_{i=1}^{i=n-1} \frac{d\theta_i}{dx_j} A^{ik}_{1h} = -A_{1k}$$

since

$$\frac{dA_{1j}}{d\frac{d\theta_i}{dx_k}} = -\frac{dA_{1k}}{d\frac{d\theta_i}{dx_j}}.$$

In all cases, then, when the above determinant does not vanish, it is equal to either A_{1h} or $-A_{1k}$. Multiplying it then by $\frac{dA_{1h}}{dx_k}$ and summing for h and k, from $h=1$ to $h=u$ and from $k=1$ to $k=u$ we have

$$\sum_{k=1}^{k=n}\sum_{h=1}^{h=n}\sum_{i=1}^{i=n-1}\frac{dA_{1h}}{dx_k}\frac{d\theta_i}{dx_j}A^{ik}_{1h}=\sum_{h=1}^{h=n}A_{1h}\frac{dA_{1h}}{dx_j}-\sum_{k=1}^{k=n}A_{1k}\frac{dA_{1j}}{dx_k}.$$

Now the function K_j may be written in the form

$$K_j=\sum_{k=1}^{k=n}A_{1k}\frac{dA_{1j}}{dx_k}-\sum_{h=1}^{h=n}A_h\frac{dA_h}{dx_j} \tag{α}$$

This differs from the second member of the last equation merely in sign. This equation may be written in the form

$$\sum_{i=1}^{i=n-1}\frac{d\theta_i}{dx_j}\left\{\sum_{k=1}^{k=n}\sum_{h=1}^{h=n}\frac{dA_{1h}}{dx_k}A^{ik}_{1h}\right\}=-K_j. \tag{β}$$

Now since

$$\frac{dA^{ik}_{11}}{dx_1}+\frac{dA^{ik}_{12}}{dx_2}+\ldots\ldots+\frac{dA^{ik}_{1n}}{dx_n}=0$$

the quantity in brackets can be written

$$\sum_{k=1}^{k=n}\frac{d}{dx_k}\left\{\sum_{h=1}^{=n}A_{1h}A^{ik}_{1h}\right\} \tag{γ}$$

Now as we have for the determination of the functions Φ a set of equations of the form

$$\Phi_1\frac{d\theta_1}{dx_j}+\Phi_2\frac{d\theta_2}{dx_j}+\ldots\ldots+\Phi_n\frac{d\theta_n}{dx_j}=\sum_{k=1}^{k=n}A_{1k}\left(\frac{dA_{1j}}{dx_k}-\frac{dA_{1k}}{dx_j}\right).$$

Now observing the equations (α) and (β) we have at once by comparing the coefficients of $\frac{d\theta_i}{dx_j}$

$$\Phi_i=-\sum_{k=1}^{k=n}\sum_{h=1}^{h=n}\frac{dA_{1h}}{dx_k}A^{ik}_{1h} \tag{53}$$

or, on taking (γ) into account

$$\Phi_i=-\sum_{k=1}^{k=n}\frac{d}{dx_k}\left\{\sum_{h=1}^{h=n}A_{1h}A^{ik}_{1h}\right\} \tag{54}$$

Now since

$$A^{ik}_{1h}=\frac{dA_{1h}}{d\frac{d\theta_i}{dx_k}}$$

it is obvious that the quantity in brackets is the sum of the differential coefficients of $2q$ with respect to $\frac{d\theta_i}{dx_k}$ and therefore finally

$$-\Phi_i = \frac{d}{dx_1}\frac{dq}{d\frac{d\theta_i}{dx_1}} + \frac{d}{dx_2}\frac{dq}{d\frac{d\theta_i}{dx_2}} + \dots + \frac{d}{dx_n}\frac{dq}{d\frac{d\theta_i}{dx_n}}. \tag{55}$$

If we multiply equations (50) by $A_{i-1,1}$, $A_{i-1,2}$, …, $A_{i-1,n}$ respectively and add, we shall obviously have

$$\Phi_i = \frac{1}{J}\{A_{i-1,1}K_1 + A_{i-1,2}K_2 + \dots + A_{i-1,n}K_n\} \tag{56}$$

a form of Φ_i from which some interesting results may be obtained. Substituting Φ_k for its value in (28) gives

$$d(U-q) = \sum_{j=1}^{j=n}\frac{dA_{1j}}{dt}dx_j + \sum_{k=1}^{k=n-1}\Phi_k d\theta_k \tag{57}$$

and from this we obtain at once

$$\begin{aligned}
\frac{d(U-q)}{dx_1} &= \frac{dA_{11}}{dt} + \Phi_1\frac{d\theta_1}{dx_1} + \Phi_2\frac{d\theta_2}{dx_1} + \dots + \Phi_{n-1}\frac{d\theta_{n-1}}{dx_1}\\
\frac{d(U-q)}{dx_2} &= \frac{dA_{12}}{dt} + \Phi_1\frac{d\theta_1}{dx_2} + \Phi_2\frac{d\theta_2}{dx_2} + \dots + \Phi_{n-1}\frac{d\theta_{n-1}}{dx_2}\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
\frac{d(U-q)}{dx_n} &= \frac{dA_{1n}}{dt} + \Phi_1\frac{d\theta_1}{dx_n} + \Phi_2\frac{d\theta_2}{dx_n} + \dots + \Phi_{n-1}\frac{d\theta_{n-1}}{dx_n}
\end{aligned} \tag{58}$$

For the case of steady motion the first terms on the right side of (58) drop out and the equations take the form

$$\frac{d(U-q)}{dx_i} = \Phi_1\frac{d\theta_1}{dx_i} + \Phi_2\frac{d\theta_2}{dx_i} + \dots + \Phi_{n-1}\frac{d\theta_{n-1}}{dx_i} \tag{59}$$

Multiplying by dx_1, dx_2, …. dx_n respectively and adding, we see that for steady motion the functions Φ are the differential coefficients with respect to θ_1, θ_2, …, θ_{n-1} of a function.

$$W = U - q = W(\theta_1\theta_2\dots\theta_{n-1})$$

i. e.

$$\Phi_1 = \frac{dW}{d\theta_1}, \quad \Phi_2 = \frac{dW}{d\theta_2}\dots\Phi_{n-1} = \frac{dW}{d\theta_{n-1}} \tag{60}$$

The operator

$$A_{11}\frac{d}{dx_1} + A_{12}\frac{d}{dx_2} + \dots + A_{1n}\frac{d}{dx_n}$$

is well known to be equivalent to

$$J\frac{d}{d\theta}.$$

If we multiply now equations (58) by $A_{11}, A_{12}, \ldots., A_{1n}$ respectively we have by virtue of this relation between the two operators

$$\frac{d(U-q)}{d\theta}, = \frac{dW}{d\theta}, = \frac{1}{J}\frac{dq}{dt} \tag{61}$$

and consequently

$$W = U - q = \int \frac{1}{J}\frac{dq}{dt}\, d\theta + \Psi(\theta_1\theta_2 \ldots . \theta_{n-1}t) \tag{62}$$

Ψ being a perfectly arbitrary functional symbol. This might be expressed differently; since θ is the only variable function we must have

$$\frac{d\theta_1}{dx_1}dx_1 + \frac{d\theta}{dx_2}dx_2 + \ldots . + \frac{d\theta}{dx_n}dx_n = d\theta$$

$$\frac{d\theta}{dx_1}dx_1 + \frac{d\theta_1}{dx_2}dx_2 + \ldots . + \frac{d\theta_1}{dx_n}dx_n = 0$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

from which follows

$$dx_1 = \frac{A_{11}d\theta}{J}$$

$$dx_2 = \frac{A_{12}d\theta}{J}.$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

Substituting the value of $d\theta$ thus obtained in the above equation we are enabled to express the integral in terms of $x_1, x_2, \ldots., x_n$. In order however to obtain the value of the integral in (62) it will be necessary to express $x_1x_2 \ldots . x_n$ in terms of $\theta, \theta_1, \theta_2, \ldots., \theta_{n-1}$. To this end consider the Jacobian

$$\Delta = \frac{d(\theta_1\theta_2 \ldots . \theta_{n-1}W)}{d(x_1x_2 \ldots . x_n)}. \tag{63}$$

The minors of Δ corresponding to $\frac{dW}{dx_1}, \frac{dW}{dx_2} \ldots .$ are the same as the minors of J corresponding to $\frac{d\vartheta}{dx_1}, \frac{d\theta}{dx_2} \ldots .$ The minor corresponding to $\frac{d\theta_i}{dx}$ is

$$\frac{dW}{dx_1}A^{i1}_{1h} + \frac{dW}{dx_2}A^{i2}_{1h} + \ldots . + \frac{dW}{dx_n}A^{in}_{1h} \tag{64}$$

which is again a Jacobian. Denote the principal minors of Δ corresponding to elements in the first row by $\Lambda_{11}, \Lambda_{12}, \ldots., \Lambda_{1n}$ and by $\Lambda_{i1}, \Lambda_{i2}, \ldots., \Lambda_{in}$ corresponding to elements in the i^{th} row. Now as we know

$$\frac{d\Lambda_{i1}}{dx_1} + \frac{d\Lambda_{i2}}{dx_2} + \ldots . + \frac{d\Lambda_{in}}{dx_n} = 0; \tag{65}$$

the quantity (64) is $= \Lambda^{ih}$, therefore (65) becomes

$$\sum_{h=1}^{h=n} \sum_{j=1}^{j=n} \frac{d}{dx_h} \left[\frac{dW}{dx_j} A^{ij}_{1h}\right] = 0. \tag{66}$$

From equations (58) we have, however,

$$\frac{dW}{dx_j} = \frac{dA_{1j}}{dt} + \Phi_1 \frac{d\theta_1}{dx_j} + \Phi_2 \frac{d\theta_2}{dx_j} + \cdots\cdots + \Phi_{n-1} \frac{d\theta_{n-1}}{dx_j}$$

Substitution of this in (66) gives a series of terms of which the first is

$$\sum_{h=1}^{h=n} \frac{d}{dx_h} \left\{ \Phi_1 \left[\frac{d\theta_1}{dx_1} a^{i1}_{1h} + \frac{d\theta_1}{dx_2} A^{i2}_{1h} + \cdots\cdots + \frac{d\theta_1}{dx_n} A^{in}_{1h}\right] \right\}$$

This is $= 0$ for all values of i other than $i = 1$. Similarly for all the terms until the i^{th}; the coefficient of Φ_i is then $= A_{1h}$. Now this substitution gives before reduction

$$0 = \sum_{h=1}^{h=n} \sum_{j=1}^{j=n} \frac{d}{dx_h} \left[A^{ij}_{1h} \left(\Phi_1 \frac{d\theta_1}{dx_j} + \Phi_2 \frac{d\theta_2}{dx_j} + \cdots\cdots + \Phi_{n-1} \frac{d\theta_{n-1}}{dx_j} + \frac{dA_{1k}}{dt} \right) \right] \tag{67}$$

on performing the above described reduction this becomes

$$\sum_{h=1}^{h=n} \frac{d}{dx_h} (A_{1h} \Phi_i) + \sum_{h=1}^{h=n} \sum_{j=1}^{j=n} \frac{d}{dx_h} \left(\frac{dA_{1j}}{dt} A^{ij}_{1h} \right) = 0 \tag{68}$$

Since the first and second minors of J, i. e. A_{1h} and A^{ij}_{1h}, all satisfy identically the equation of continuity, (68) may be written in the form

$$\begin{aligned}
&A_{11} \frac{d\Phi_1}{dx_1} + A_{12} \frac{d\Phi_1}{dx_2} + \cdots\cdots + A_{1n} \frac{d\Phi_1}{dx_n} + \Theta_1 = 0 \\
&A_{11} \frac{d\Phi_2}{dx_1} + A_{12} \frac{d\Phi_2}{dx_2} + \cdots\cdots + A_{1n} \frac{d\Phi_2}{dx_n} + \Theta_2 = 0 \\
&\cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot \\
&A_{11} \frac{d\Phi_{n-1}}{dx_1} + A_{12} \frac{d\Phi_{n-1}}{dx_2} + \cdots\cdots + A_{1n} \frac{d\Phi_{n-1}}{dx_n} + \Theta = 0
\end{aligned} \tag{69}$$

in which

$$\Theta_i = \sum_{h=1}^{h=n} \sum_{j=1}^{j=n} A^{ij}_{h} \frac{d}{dt} \frac{dA_{1j}}{dx_h} \tag{70}$$

Again, by aid of (53) we can replace (69) by

$$\begin{aligned}
&A_{11} \frac{d\Phi_1}{dx_1} + A_{12} \frac{d\Phi_1}{dx_2} + \cdots\cdots + A_{1n} \frac{d\Phi_1}{dx_n} + \frac{d\Phi_1}{dt} = \chi_1 \\
&A_{11} \frac{d\Phi_2}{dx_1} + A_{12} \frac{d\Phi_2}{dx_2} + \cdots\cdots + A_{1n} \frac{d\Phi_2}{dx_n} + \frac{d\Phi_2}{dt} = \chi_2 \\
&\cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot \\
&A_{11} \frac{d\Phi_{n-1}}{dx_1} + A_{12} \frac{d\Phi_{n-1}}{dx_2} + \cdots\cdots + A_{1n} \frac{d\Phi_{n-1}}{dx_n} + \frac{d\Phi_{n-1}}{dt} = \chi_{n-1}
\end{aligned} \tag{71}$$

in which

$$\chi_i = \sum_{k=1}^{k=n} \sum_{h=1}^{h=n} \frac{dA^{1h}}{dx_k} \frac{dA^{ik}_{1h}}{dt} \tag{72}$$

The modifications undergone by (69) and (71) in the case of steady motion are obvious and need no remark.

The equations

$$\frac{dx_1}{dt} = u_1, \qquad \frac{dx_2}{dt} = u_2, \text{ etc.}$$

have been replaced by

$$\frac{dx_1}{dt} = A_{11}, \qquad \frac{dx_2}{dt} = A_{12}, \text{ etc.} \tag{73}$$

If we have

$$\frac{d\theta_k}{dt} = 0$$

then $\theta_k =$ const. is an integral of these equations, and similarly for any other of the functions θ. If

$$\chi_k = 0$$

then

$$\frac{d\Phi_k}{dt} = A_1 \frac{d\Phi_k}{dx_1} + A_2 \frac{d\Phi_k}{dx_2} + \cdots + \frac{d\Phi_k}{dx_n} + \frac{d\Phi_k}{dt} = \chi_k = 0$$

and in this case

$$\Phi_k = \text{const.}$$

is also an integral of (73). If all of the functions θ_j are independent of t, we shall have

$$\theta_1 = \text{const.}, \quad \theta_2 = \text{const.} \ldots\ldots \theta_{n-1} = \text{const.}$$

as integrals of (73). Suppose in equations (10) that the functions u were

$$u_0 \, u_1 \, u_2 \ldots\ldots u_{2n}$$

then it is known (see article of Professor Tanner's above referred to) that

$$u_0 dx_0 + u_1 dx_1 + \cdots + u_{2n} dx_{2n}$$

can always be reduced to the form

$$dM_0 + L_1 dM_1 + L_2 dM_2 + \cdots + L_n dM_n$$

and

$$u_k = \frac{dM_0}{dx_k} + L_1 \frac{dM_1}{dx_k} + \cdots + L_n \frac{dM_n}{dx_k} \tag{74}$$

In this case we should also have

$$d(U - q) = \sum_{k=0}^{k=2n} \frac{dn_k}{dt} dx_k + \tfrac{1}{2} \sum_{j=0}^{j=2n} \sum_{k=0}^{k=2n} \left(\frac{dn_k}{dx_j} - \frac{dn_j}{dx_k} \right) (u_j dx_k - u_k dx_j) \tag{75}$$

Now clearly

$$\frac{du_k}{dx_j} - \frac{du_j}{dx_k} = \sum_{r=0}^{r=2n} \begin{Vmatrix} \dfrac{dL_r}{dx_j} & \dfrac{dL_r}{dx_k} \\[2ex] \dfrac{dM_r}{dx_j} & \dfrac{dM_r}{dx_k} \end{Vmatrix}$$

The double sum in (75) thus becomes

$$= \sum_{r=0}^{r=2n} \begin{Vmatrix} \displaystyle\sum_{\rho=0}^{\rho=2n} u_\rho \frac{dL_r}{dx_\rho} & \displaystyle\sum_{\rho=0}^{\rho=2n} \frac{dL_r}{dx_\rho} dx_\rho \\[2ex] \displaystyle\sum_{\rho=0}^{\rho=2n} u_\rho \frac{dM_r}{dx_\rho} & \displaystyle\sum_{\rho=0}^{\rho=2n} \frac{dM_r}{dx_\rho} dx_\rho \end{Vmatrix}$$

and since

$$\frac{dx_j}{dt} = u_j$$

this is

$$= \sum_{r=0}^{r=2n} \left\{ dM_r \begin{Vmatrix} \dfrac{D}{dt} & \dfrac{d}{dt} \\[2ex] L_r & L_r \end{Vmatrix} - dL_r \begin{Vmatrix} \dfrac{D}{dt} & \dfrac{d}{dt} \\[2ex] M_r & M_r \end{Vmatrix} \right\}$$

Substituting this in (75) we have, after some reductions,

$$d\left[U - q - \frac{dM}{dt} - \sum_{r=0}^{r=2n} L_r \frac{dM_r}{dt}\right] = \sum_{r=0}^{r=2n} \left(\frac{dL_r}{dt} dM_r - \frac{dM_r}{dt} dL_r\right) \tag{76}$$

The equations

$$\begin{aligned} \frac{dU}{dx_0} &= \frac{du_0}{dt} + u_0 \frac{du_0}{dx_0} + u_1 \frac{du_0}{dx_1} + \cdots + u_{2n} \frac{du_0}{dx_{2n}} \\ \frac{dU}{dx_1} &= \frac{du_1}{dt} + u_0 \frac{du_1}{dx_0} + u_1 \frac{du_1}{dx_1} + \cdots + u_{2n} \frac{du_1}{dx_{2n}} \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \frac{dU}{dx_{2n}} &= \frac{du_{2n}}{dt} + u_0 \frac{du_{2n}}{dx_0} + u_1 \frac{du_{2n}}{dx_1} + \cdots + u_{2n} \frac{du_{2n}}{dx_{2n}} \end{aligned} \tag{77}$$

and

$$\frac{du_0}{dx_0} + \frac{du_1}{dx_1} + \frac{du_2}{dx_2} + \cdots + \frac{du_{2n}}{dx_{2n}} = 0 \tag{78}$$

are thus replaced by a set of the form

$$\begin{aligned} \frac{dL_r}{dt} + u_0 \frac{dL_r}{dx_0} + u_1 \frac{dL_r}{dx_1} + \cdots + u_{2n} \frac{dL_r}{dx_{2n}} &= \frac{d\Omega}{dM_r} \\ \frac{dM_r}{dt} + u_0 \frac{dM_r}{dx_0} + u_1 \frac{dM_r}{dx_1} + \cdots + u_{2n} \frac{dM_r}{dx_{2n}} &= -\frac{d\Omega}{dL_r} \end{aligned} \tag{79}$$

(78) remaining unchanged. In (79) r has, of course, all values from 0 to $2n$, and Ω is an arbitrary function of the L's, M's, and t. For U we have now

$$U = \frac{dM_0}{dt} + \sum_{\rho=1}^{\rho=2n} L_\rho \frac{dM_\rho}{dt} + \tfrac{1}{2} \sum_{k=0}^{k=2n} \left\{ \frac{dM_0}{dx_k} + \sum_{\rho=1}^{\rho=2n} L_\rho \frac{dM_\rho}{dx_k} \right\} + \Omega \tag{80}$$

Equations (59), which correspond to the case of steady motion, give us, on multiplication by $A_{i-1,1}$, $A_{i-1,2}$, ... $A_{i-1,n}$ and addition of the results,

$$\Phi_i = \frac{1}{J}\left(A_{i-1,1} \frac{dW}{dx_1} + A_{i-1,2} \frac{dW}{dx_2} + \cdots\cdot + A_{i-1,n} \frac{dW}{dx_n} \right) \tag{81}$$

expressions similar to (56) for the determination of the functions Φ.

We have seen that if we have

$$\frac{D\theta_k}{dt} = 0 \tag{82}$$

then θ_k is an integral of equations (73) or of

$$\frac{dx_1}{dt} = u_1, \frac{dx_2}{dt} = u_2 \cdots\cdot \frac{dx_n}{dt} = u_n \tag{83}$$

In order now to obtain an idea of the nature of the functions $\theta, \theta_1 \ldots\ldots \theta_{n-1}$, consider a function of the $n + 1$ variables $x_1, x_2 \ldots\ldots x_{n,t}$, which denote by f, i. e.

$$f \equiv f(x_1, x_2 \ldots\ldots x_{n,t}).$$

This function f which satisfies say

$$f = \text{const.}$$

sustains the same relation to the space of n dimensions as a function of the three variables x, y, z, and also of t sustains to ordinary space, so for want of a better term I shall speak of it as a hypersurface. Now suppose the hypersurface f to be composed always of the same particles, it must obviously satisfy the differential equation

$$\frac{df}{dt} + u_1 \frac{df}{dx_1} + u_2 \frac{df}{dx_2} + \cdots\cdot + u_n \frac{df}{dx_n} = 0 \tag{84}$$

(84) has as we know n independent integrals which we may denote by

$$\theta, \theta_1, \theta_2 \ldots\ldots \theta_{n-1}$$

The equations

$$\theta = \text{const.}, \quad \theta_1 = \text{const.}, \text{ etc.}$$

are, therefore, the equations of moving hypersurfaces which contain at all times the same fluid particles. The particles which are at any time on the intersection of any $n - 1$ of these hypersurfaces will remain on that intersection. As before we denote the Jacobian of the functions $\theta, \theta_1, \theta_2, \ldots\ldots \theta_{n-1}$ by J. So far as it has been convenient from considerations of symmetry to write θ, θ_1, $\theta_2 \ldots \theta_{n-1}$ as our functions, for the same reason it will now be more convenient to write θ_1, θ_2, $\theta_3 \ldots \theta_n$. In all that precedes, then, the function θ is identical with the new θ_1, and in general θ_{i-1} (old notation) $= \theta_i$ (new notation). The minors of J corresponding to

$$\frac{d\theta_1}{dx_1}\, \frac{d\theta_1}{dx_2} \ldots \frac{d\theta_1}{dx_n}$$

are A_{11}, $A_{12} \ldots A_{1n}$ respectively; and similarly for the other rows.

If we solve the equations

$$\begin{array}{l}
\dfrac{d\theta_1}{dt} + u_1 \dfrac{d\theta_1}{dx_1} + u_2 \dfrac{d\theta_1}{dx_2} + \ldots + u_n \dfrac{d\theta_1}{dx_n} = 0 \\
\dfrac{d\theta_2}{dt} + u_1 \dfrac{d\theta_2}{dx_1} + u_2 \dfrac{d\theta_2}{dx_2} + \ldots + u_n \dfrac{d\theta_2}{dx_n} = 0 \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\dfrac{d\theta_n}{dt} + u_1 \dfrac{d\theta_n}{dx_1} + u_2 \dfrac{d\theta_n}{dx_2} + \ldots + u_n \dfrac{d\theta_n}{dx_n} = 0
\end{array} \tag{85}$$

we find

$$\begin{array}{l}
Ju_1 = - \left\{ \dfrac{d\theta_1}{dt} A_{11} + \dfrac{d\theta_2}{dt} A_{21} + \ldots + \dfrac{d\theta_n}{dt} A^{n1} \right\} \\
Ju_2 = - \left\{ \dfrac{d\theta_1}{dt} A_{12} + \dfrac{d\theta_2}{dt} A_{22} + \ldots + \dfrac{d\theta_n}{dt} A_{n2} \right\} \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
Ju_n = - \left\{ \dfrac{d\theta_1}{dt} A_{1n} + \dfrac{d\theta_2}{dt} A_{2n} + \ldots + \dfrac{d\theta_n}{dt} A_{nn} \right\}
\end{array} \tag{86}$$

Substituting these values of u in

$$\frac{du_1}{dx_1} + \frac{du_2}{dx_2} + \ldots + \frac{du_n}{dx_n} = 0$$

and we find by the rule for differentiating a determinant

$$u_1 \frac{dJ}{dx_1} + u_2 \frac{dJ}{dx_2} + \ldots + u_n \frac{dJ}{dx_n} = - \frac{dJ}{dt} \tag{87}$$

or J satisfies the equation

$$\frac{DJ}{dt} = 0. \tag{88}$$

It would be interesting to determine the effect of the operator $\frac{D}{dt}$ upon the various minors of J, but it is sufficient for the present purpose to find the effect produced upon the first minors by this operator. The method of doing this is, in the case of

A_{11}, to erase the first of equations (85) and solve the remaining $n-1$ equations for $u_2, u_3, \ldots, u_n$. We have after doing this

$$\left.\begin{aligned} A_{11}u_2 &= -\left\{\left[\frac{d\theta_2}{dt}+u_1\frac{d\theta_2}{dx_1}\right]A^{11}_{11}+\left[\frac{d\theta_3}{dt}+u_1\frac{d\theta_3}{dx_1}\right]A^{11}_{21}+\ldots+\left[\frac{d\theta_n}{dt}+u_1\frac{d\theta_n}{dx_1}\right]A^{11}_{n-1,1}\right\} \\ A_{11}u_3 &= -\left\{\left[\frac{d\theta_2}{dt}+u_1\frac{d\theta_2}{dx_1}\right]A^{21}_{11}+\left[\frac{d\theta_3}{dt}+u_1\frac{d\theta_3}{dx_1}\right]A^{21}_{21}+\ldots+\left[\frac{d\theta_n}{dt}+u_1\frac{d\theta_n}{dx_1}\right]A^{21}_{n-1,1}\right\} \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \quad (89)$$

Differentiate these with respect to $x_2, x_3, \ldots x_n$ and add and we readily find

$$\frac{DA_{11}}{dt} = A_{11}\frac{du_1}{dx_1} + A_{12}\frac{du_1}{dx_2} + \ldots + A^{1n}\frac{du_1}{dx_n}$$

and in like manner

$$\left.\begin{aligned} \frac{DA_{12}}{dt} &= A_{11}\frac{du_2}{dx_1} + A_{12}\frac{du_2}{dx_2} + \ldots + A_{1n}\frac{du_2}{dx_n} \\ \frac{DA_{13}}{dt} &= A_{11}\frac{du_3}{dx_1} + A_{12}\frac{du_3}{dx_2} + \ldots + A_{1n}\frac{du_3}{dx_n} \\ &\ldots\ldots\ldots\ldots \\ \frac{DA_{1n}}{dt} &= A_{11}\frac{du_n}{dx_1} + A_{12}\frac{du_n}{dx_2} + \ldots + A_{1n}\frac{du_n}{dx_n} \end{aligned}\right. \quad (90)$$

If in these equations we substitute for A_{1i} the other principal minors $A_{2i}, A_{3i}, \ldots A_{ni}$ (i having all values from $i = 1$ to $i = u$ in all of these) the equations will still hold. I do not write the proof of this out in full, as it is extremely simple and yet rather tedious.

If we combine the equations $\theta = \text{const.}$ in groups of $n-1$ each, we shall of course obtain the n lines representing the intersections of n hypersurfaces represented by the equations $\theta = \text{const.}$ Call these lines $s_1, s_2, \ldots s_n$; and denote by $ds_1, ds_2, \ldots ds_n$ the n sides of an element of fluid occupying the space between the hypersurfaces

$$\theta_1 = \alpha_1, \qquad \theta_2 = \alpha_2 \ldots \theta_n = \alpha_n$$

and

$$\theta_1 = \alpha_1 + d\alpha_1, \qquad \theta_2 = \alpha_2 + d\alpha_2 \ldots \theta_n = \alpha_n + d\alpha_n.$$

We must have now

$$\left.\begin{aligned} \alpha_1 &= \theta_1(x_1, x_2 \ldots x_n, t) \\ \alpha_1 + d\alpha_1 &= \theta_1(x_1 + \lambda_{11}ds_1 + x_2 + \lambda_{12}ds_2 \ldots x_n + \lambda_{1n}ds_n, t) \end{aligned}\right\} \quad (91)$$

the quantities $\lambda_{11}, \lambda_{12}, \ldots \lambda_{1n}$ denoting the 'direction cosines' of ds_1. Similar equations of course hold for $\alpha_2, \alpha_3, \ldots \alpha_n$. By Taylor's theorem we now obtain

$$\begin{aligned} d\alpha_1 &= \left\{ \lambda_{11} \frac{d\theta_1}{dx_1} + \lambda_{12} \frac{d\theta_1}{dx_2} + \ldots\ldots + \lambda_{1n} \frac{d\theta_1}{dx_n} \right\} ds_1 \\ d\alpha_2 &= \left\{ \lambda_{21} \frac{d\theta_2}{dx_1} + \lambda_{22} \frac{d\theta_2}{dx_2} + \ldots\ldots + \lambda_{2n} \frac{d\theta_2}{dx_n} \right\} ds_2 \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ d\alpha_n &= \left\{ \lambda_{n1} \frac{d\theta_n}{dx_1} + \lambda_{n2} \frac{d\theta_n}{dx_2} + \ldots\ldots + \lambda_{nn} \frac{d\theta_n}{dx_n} \right\} ds_n \end{aligned} \tag{92}$$

The values of the λ's are known to be

$$\lambda_{11} = \frac{A_{11}}{2q}, \qquad \lambda_{12} = \frac{A_{12}}{2q}, \text{ etc.} \ldots\ldots \tag{93}$$

where

$$2q = A_{11}^2 + A_{12}^2 + \ldots\ldots + A_{1n}^2. \tag{94}$$

Similar expressions hold for λ_{ji} ($j = 2, 3 \ldots\ldots n$ and $i = 1, 2 \ldots\ldots n$). The quantity $2q$ can be expressed by a determinant viz.

$$2q = \left\| \begin{matrix} c_{22} & c_{23} & \ldots\ldots & c^{2n} \\ c_{32} & c_{33} & \ldots\ldots & c_{3n} \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ c_{n2} & c_{n3} & \ldots\ldots & c^{nn} \end{matrix} \right\| \tag{95}$$

in which

$$\begin{gathered} c_{22} = \sum_j \left(\frac{d\theta_2}{dx_j} \right)^2, \qquad c_{33} = \sum_j \left(\frac{d\theta_3}{dx_j} \right)^2, \text{ etc.} \ldots\ldots \\ c_{23} = \sum_j \frac{d\theta_2}{dx_j} \frac{d\theta_3}{dx_j} \; c_{24} = \sum_j \frac{d\theta_2}{dx_j} \frac{d\theta_4}{dx_j} \text{ etc.} \ldots\ldots \end{gathered} \tag{96}$$

j taking all values from $j = 1$ to $j = n$; and $c_{ik} = c_{ki}$. Substituting the values of $\lambda_{11}, \lambda_{12}, \ldots\ldots, \lambda_{1n}$ from equations (93) in the first of equations (92) and we find at once

$$ds_1 = \frac{2q, d\alpha_1}{J}, = 2q \frac{d\theta_1}{J}; \tag{97}$$

similar expressions are to be found for $ds_2, ds_3, \ldots\ldots ds_n$. Denote the projections of ds_1 on the axis by

$$dx_1^{(1)}, dx_2^{(1)}, \ldots\ldots dx_n^{(1)};$$

then from what precedes

$$u_1 dx_1^{(1)} + u_2 dx_2^{(1)} + \ldots\ldots + u_n dx_n^{(1)} = \frac{d\theta_1}{J} \left\| \begin{matrix} u_1, & u_2, & \ldots\ldots, & u_n \\ \frac{d\theta_2}{dx_1}, & \frac{d\theta_2}{dx_2}, & \ldots\ldots, & \frac{d\theta_2}{dx_n} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \frac{d\theta_n}{dx_1}, & \frac{d\theta_n}{dx_2}, & \ldots\ldots, & \frac{d\theta_n}{dx_n} \end{matrix} \right\| \tag{98}$$

with similar expressions resulting from the projection of $ds_2, ds_3, \ldots . ds_n$ on the axes. Comparing these equations with (39) and writing

$$U + q = W \tag{99}$$

gives

$$\frac{D}{dt}(u_1 A_{11} + u_2 A_{12} + \ldots . + u_n A_{1n}) \frac{d\theta_1}{J} = dW \tag{100}$$

and by (88)

$$\frac{D}{dt}(u_1 A_{11} + u_2 A_{12} + \ldots . + u_n A_{1n}) = J \frac{dW}{d\theta_1} \tag{101}$$

and in like manner

$$\frac{D}{dt}(u_1 A_{21} + u_2 A_{22} + \ldots . + u_n A_{2n}) = J \frac{dW}{d\theta_2}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\frac{D}{dt}(u_1 A_{n1} + u_2 A_{n2} + \ldots . + u_n A_{nn}) = J \frac{dW}{d\theta_n}$$

Assume a function P such that

$$W = \frac{DP}{dt} \tag{102}$$

then since

$$A_{11}\frac{d}{dx_1} + A_{12}\frac{d}{dx_2} + \ldots . + A_{1n}\frac{d}{dx_n} = J\frac{d}{d\theta_1} \text{ etc.} \tag{103}$$

we have

$$\begin{aligned} u_1 A_{11} + u_2 A_{12} + \ldots . + u_n A_{1n} &= J \left\{ \frac{dP}{d\theta_1} + \psi_1 \right\} \\ u_1 A_{21} + u_2 A_{22} + \ldots . + u_n A_{2n} &= J \left\{ \frac{dP}{d\theta_2} + \psi_2 \right\} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad &\cdot \quad \cdot \quad \cdot \\ u_1 A_{n1} + u_2 A_{n2} + \ldots . + u_n A_{nn} &= J \left\{ \frac{dP}{d\theta_n} + \psi_n \right\} \end{aligned} \tag{104}$$

The functions ψ contain $\theta_1, \theta_2, \ldots . \theta_n$ but not t. Solution of (104) gives

$$u_1 = \frac{d\theta_1}{dx_1}\left\{\frac{dP}{d\theta_1} + \psi_1\right\} + \frac{d\theta_2}{dx_1}\left\{\frac{dP}{d\theta_2} + \psi_2\right\} + \ldots . + \frac{d\theta_n}{dx_1}\left\{\frac{dP}{d\theta_n} + \psi_n\right\} \text{ etc.}$$

bearing in mind that

$$\| A_{ik} \| = J^{n-1} \tag{105}$$

These equations for $u_1, u_2 \ldots . u_n$ may be written in the form

$$\begin{aligned} u_1 &= \frac{dP}{dx_1} + \psi_1 \frac{d\theta_1}{dx_1} + \psi_2 \frac{d\theta_2}{dx_1} + \ldots . + \psi_n \frac{d\theta_n}{dx_1} \\ u_2 &= \frac{dP}{dx_2} + \psi_1 \frac{d\theta_1}{dx_2} + \psi_2 \frac{d\theta_2}{dx_2} + \ldots . + \psi_n \frac{d\theta_n}{dx_2} \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ u_n &= \frac{dP}{dx_n} + \psi_1 \frac{d\theta_1}{dx_n} + \psi_2 \frac{d\theta_2}{dx_n} + \ldots . + \psi_n \frac{d\theta_n}{dx_n} \end{aligned} \qquad (106)$$

and from these follows

$$u_1 dx_1 + u_2 dx_2 + \ldots . + u_n dx_n = dP + \psi_1 d\theta_2 + \psi_2 d\theta_2 + \ldots . + \psi_n d\theta_n \qquad (107)$$

This result may be obtained in another manner which will bring into evidence some interesting properties of the functions with which we are dealing. The method to be employed I shall merely indicate, as it is similar to that employed in obtaining equations (61) and (62). The original equations of motion are

$$\frac{Du_1}{dt} = \frac{dU}{dx_1}, \qquad \frac{Du_2}{dt} = \frac{dU}{dx_2} \text{ etc.} \ldots . \qquad (108)$$

now

$$\frac{d}{dx_i} = \frac{d\theta_1}{dx_i}\frac{d}{d\theta_1} + \frac{d\theta_2}{dx_i}\frac{d}{d\theta_2} + \ldots . + \frac{d\theta_n}{dx_i}\frac{d}{d\theta_n} \qquad (109)$$

and consequently

$$\begin{aligned} \frac{Du_1}{dt} &= \frac{d\theta_1}{dx_1}\frac{dU}{d\theta_1} + \frac{d\theta_2}{dx_1}\frac{dU}{d\theta_2} + \ldots . + \frac{d\theta_n}{dx_1}\frac{dU}{d\theta_n} \\ \frac{Du_2}{dt} &= \frac{d\theta_1}{dx_2}\frac{dU}{d\theta_1} + \frac{d\theta_2}{dx_2}\frac{dU}{d\theta_2} + \ldots . + \frac{d\theta_n}{dx_2}\frac{dU}{d\theta_n} \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \frac{Du_n}{dt} &= \frac{d\theta_1}{dx_n}\frac{dU}{d\theta_1} + \frac{d\theta_2}{dx_n}\frac{dU}{d\theta_2} + \ldots . + \frac{d\theta_n}{dx_n}\frac{dU}{d\theta_n} \end{aligned} \qquad (110)$$

Multiply these by $A_{11}, A_{12}, \ldots ., A_{1n}$ respectively; then by $A_{21}, A_{22}, \ldots ., A_{2n}$ respectively, etc.; add in each case and we shall obtain

$$\left\| \begin{matrix} \frac{Du_1}{dt}, & \frac{Du_2}{dt}, & \ldots ., & \frac{Du_n}{dt} \\ \frac{d\theta_2}{dx_1}, & \frac{d\theta_2}{dx_2}, & \ldots ., & \frac{d\theta_2}{dx_n} \\ \vdots & & & \\ \frac{d\theta_n}{dx_1}, & \frac{d\theta_n}{dx_2}, & \ldots ., & \frac{d\theta_n}{dx_n} \end{matrix} \right\| = J \frac{dU}{d\theta_1}, \text{ etc.} \qquad (111)$$

Now we have found that

$$\frac{DA_{11}}{dt} = A_{11}\frac{du_1}{dx_1} + A_{12}\frac{du_1}{dx_2} + \ldots . + A_{1n}\frac{du_1}{dx_n}$$

Using this equation, and also equation (103) and we are enabled to write at once

$$\begin{aligned} \frac{DA_{11}}{dt} &= J\frac{du_1}{d\theta_1} \\ \frac{DA_{12}}{dt} &= J\frac{du_2}{d\theta_1} \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \frac{DA_{1n}}{dt} &= J\frac{du_n}{d\theta_1} \end{aligned} \qquad (112)$$

Multiplying these by $u_1, u_2, \ldots . u_n$ respectively and adding the sum to (111) we obtain

$$\frac{D}{dt}\left\|\begin{matrix} u_1\ , & u_2\ , & \ldots ., & u_n \\ \frac{d\theta_2}{dx_1}, & \frac{d\theta_2}{dx_2}, & \ldots ., & \frac{d\theta_2}{dx_n} \\ \vdots & & & \\ \frac{d\theta_n}{dx_1}, & \frac{d\theta_n}{dx_2}, & \ldots ., & \frac{d\theta_n}{dx_n} \end{matrix}\right\| = J\frac{dW}{d\theta_1}$$

there are of course $n - 1$ other equations of this form which can be immediately written down. Since

$$\frac{DJ}{dt} = 0$$

we have, if we write $P = \frac{DW}{dt}$ integrate and solve the resulting equations for $u_1, u_2, \ldots . u_n$ the same expressions as those marked (106).

NOTE.—I desire here to call attention to the paper published by Professor Rowland in the American Journal of Mathematics, Vol. III. No. 3, in which a full account of the so-called higher orders of motion is contained. (I may also mention that I am informed by Professor Rowland that he gave a brief account of these motions at the meeting of the Scientific Association of the Johns Hopkins University on October 6th).

The existence of these motions was discovered at about the same time independently by Professor Rowland and myself. He was led to consider them while treating of the equations of electromagnetism, and their connection with fluid motion. I was led to similar results by the study of certain possible cases of steady motion in viscous fluids, an account of which will be found in the Journal of the Franklin Institute for October, 1880, and also in the Philosophical Magazine for November, 1880.

I had devised no simple means of determining the lower orders of motion in terms of the higher orders, but had merely derived the higher orders from the lower by obvious processes of differentiation. In Professor Rowland's paper an elegant solution of the much more difficult problem is given. I may further say that up to the time of my becoming acquainted with Professor Rowland's research I had found no physical interpretation of these quantities so that the entire credit for the development of the theory of these motions is due solely to Professor Rowland. In the present paper the quantities corresponding to the higher orders of motion are denoted by the letter η with different accents and subscripts. My work on the subject of the higher orders of motion is to be found in the paper quoted at the beginning of this article, "General Properties of the Equations of Steady Motion." I believe that Professor Rowland has made no use of the integrating factors which I have there introduced. This part of the subject I hope to go into quite fully in Part II. of the present paper.

WASHINGTON, D. C.
January 2, 1881.